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Differential Geometry and its Applications 18 (2003) 55–78

APPLICATIONS

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Family of Cayley transforms of a homogeneous Siegel domain parametrized by admissible linear forms

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Received 19 March 2001; received in revised form 21 June 2001

Communicated by S. Gindikin

Abstract

In this paper we introduce a family of Cayley transforms of a homogeneous Siegel domain D. Each of these transforms is birational and maps D biholomorphically onto a bounded domain in a complex Euclidean vector space.

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MSC: 22E25; 32M10

Keywords: Homogeneous Siegel domain; Normal j-algebra; Cayley transform

1. Introduction

This paper is a continuation of our study of Cayley transforms of a homogeneous Siegel domain D published as [10]. The Cayley transform studied in the previous paper is defined through a function which yields the Bergman kernel of D, whereas Penney's in [14] through the characteristic function of the cone Ω in the defining data of D. On the other hand, as our study [13] of harmonicity property of the Poisson kernel of D goes on, it has turned out that another Cayley transform, the one associated to the Szegö kernel, necessarily appears. If D is symmetric and irreducible, it is known (see [8, Proposition 5.3] for example) that the Bergman kernel is a power of the Szegö kernel up to a positive constant multiple, so that the newly introduced Cayley transform is not essentially different from the previous one studied in [10]. However, if D is not a quasisymmetric type II domain, this new one as well as the preceding two differ slightly from each other. Since still another Cayley transform might come out in future studies of

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PII: S0926-2245(02)00098-0

analysis on non-symmetric Siegel domains, it is a matter of course to try to construct a family of Cayley transforms that includes the above three. This is the first motivation for writing the present paper.

To explain how we parametrize our family of Cayley transforms, let us fix our notation. Let G be the split solvable Lie group acting simply transitively on D. As is known [15], the Lie algebra $\mathfrak{g}$ of G has a structure of normal j-algebra. In particular, we have an integrable almost complex structure J on $\mathfrak{g}$, and there is a linear form $\omega \in \mathfrak{g}^*$ such that the bilinear form $\langle [Jx, y], \omega \rangle$ defines a J-invariant inner product on $\mathfrak{g}$. Such linear forms ω are said to be *admissible*. Basic structure theory of $\mathfrak{g}$ tells us that $\mathfrak{g}$ always contains a product of r copies of the $(ax+b)$-algebra, where r is the rank of the normal j-algebra $\mathfrak{g}$. Thus there are $2r$ elements $H_1, \ldots, H_r$ and $E_1, \ldots, E_r$ such that $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \ldots, \alpha_r$ and $E_1^*, \ldots, E_r^*$ be the dual basis. We know that $\mathfrak{a} := \sum_1^r \mathbb{R} H_j$ gives rise to a root space decomposition of $\mathfrak{g}$, in which $\mathbb{R}E_j$ coincides with the α_j-root space. We extend each E_j^* to the whole $\mathfrak{g}$ by zero extension on $\mathfrak{a}$ and on the root spaces other than $\mathbb{R}E_j$. Then, admissible linear forms are essentially of the form $E_{\mathbf{s}}^* := s_1 E_1^* + \cdots + s_r E_r^*$ ($\mathbf{s} = (s_1, \ldots, s_r)$) with $s_j > 0$ for all j (see Proposition 3.4 for the precise statement). To an admissible $E_{\mathbf{s}}^*$, there corresponds a strictly log-convex function $\Delta_{-\mathbf{s}}$ on the cone Ω that tends to ∞ on the boundary $\partial\Omega$ (see (3.16)). We note that the characteristic function of Ω and the function that yields the Bergman kernel or the Szegö kernel are of this type of $\Delta_{-\mathbf{s}}$ with specific parameter $\mathbf{s}$ up to positive constant multiples. Now the *pseudoinverse* $\mathcal{I}_{\mathbf{s}}(x)$ for $x \in \Omega$ is defined to be $-\nabla \log \Delta_{-\mathbf{s}}(x)$, minus of the gradient of $\log \Delta_{-\mathbf{s}}(x)$, just as Vinberg's x^* in [18]. Once we have the pseudoinverse map $\mathcal{I}_{\mathbf{s}}$, our Cayley transform $\mathcal{C}_{\mathbf{s}}$ is defined exactly in the same way as in the previous paper [10] (see (4.2)). Birationality and biholomorphy of $\mathcal{C}_{\mathbf{s}}$ can be shown analogously.

The second motivation for writing this paper is that we have a straightforward proof for the boundedness of the images $\mathcal{C}_{\mathbf{s}}(D)$ (Theorem 4.17). Although the proof presented here is still through an induction on the rank of normal j-algebra, it differs from the previous one presented in [10] which traced Penney's argument [14] in that it is not necessary to have the truth of the statement for symmetric domains shown separately. We remark that for general $\mathbf{s} > 0$, the image $\mathcal{C}_{\mathbf{s}}(D)$ for symmetric D is not the standard Harish-Chandra realization of a Hermitian symmetric space. This means that the previous way of proof does not work smoothly for general $\mathbf{s} > 0$.

For tube domains, definition (4.1) of our Cayley transform says that it suffices to estimate $\mathcal{I}_{\mathbf{s}}(x + E + iy)$ for $x \in \Omega$ and $y \in V$. This is done in Section 4.3 (Theorem 4.10). Our method of estimation makes use of the equivariance property (3.14) of $\mathcal{I}_{\mathbf{s}}$, which reduces the estimation to operators of adjoint actions (see Propositions 4.5 and 4.16). For type II domains, we need a little more analysis that generalizes the proof of [2, Theorem 2.8] done for the case of quasisymmetric domains. Note that we no longer have Jordan algebra structure in the ambient vector space of Ω.

In the last section, Section 4, we describe our family of Cayley transforms for Pjatetskii-Shapiro's 4-dimensional non-quasisymmetric Siegel domain. It is a non-symmetric type II domain over a symmetric cone. The reason for treating this example is to point out that seemingly the most natural Cayley transform as defined by Geatti [6] need not always be the most suitable one for problems that one deals with. This phenomenon seems to agree well with the common understanding that non-quasisymmetric Siegel domains do not have a canonical bounded model. We hope that our family of Cayley transforms provides sufficiently many bounded models to pursue further studies of analysis on Siegel domains which are not necessarily symmetric.

2. Preliminaries

2.1. Normal j-algebras

A triple $(\mathfrak{g}, J, \omega)$ of a split solvable Lie algebra $\mathfrak{g}$, a linear operator J on $\mathfrak{g}$ with $J^2 = -I$ and a linear form ω on $\mathfrak{g}$ is called a *normal j-algebra* if the following two conditions are satisfied:

$$[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad (\text{for all } x, y \in \mathfrak{g}), \tag{2.1}$$

$$\langle x \mid y \rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines a } J\text{-invariant inner product on } \mathfrak{g}. \tag{2.2}$$

Linear forms ω on $\mathfrak{g}$ satisfying (2.2) are said to be *admissible*, and the set of admissible linear forms will be described in the next section (Proposition 3.4). Here we fix one admissible linear form to present basic structural facts about normal j-algebras. Our references are [15] and [17] (see also [16]). Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. Let $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ relative to the inner product $\langle \cdot \mid \cdot \rangle_\omega$. Then $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$. We know that $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that every operator in $\mathrm{ad}(\mathfrak{a})$ is diagonalizable on $\mathfrak{g}$. Thus we have a simultaneous eigenspace decomposition $\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$, where

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n};\ [h, x] = \langle h, \alpha \rangle\, x \text{ for all } h \in \mathfrak{a}\}.$$

The dimension $r := \dim \mathfrak{a}$ is called the *rank* of the normal j-algebra. One can choose a basis $H_1, \ldots, H_r$ of $\mathfrak{a}$ such that if we set $E_j := -JH_j$, then $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$. Then elements of Δ, which we call the *roots* of $\mathfrak{g}$, are of the following form (not all possibilities need occur):

$$\begin{aligned} &\tfrac{1}{2}(\alpha_m + \alpha_k) \quad (1 \leqslant k < m \leqslant r), && \tfrac{1}{2}(\alpha_m - \alpha_k) \quad (1 \leqslant k < m \leqslant r), \\ &\tfrac{1}{2}\alpha_k \quad (1 \leqslant k \leqslant r), && \alpha_k \quad (1 \leqslant k \leqslant r). \end{aligned} \tag{2.3}$$

We note that $\mathfrak{n}_{\alpha_k} = \mathbb{R}E_k$ and that if α, β are distinct roots, then $\mathfrak{n}_\alpha$ is orthogonal to $\mathfrak{n}_\beta$. Put

$$H := H_1 + \cdots + H_r, \qquad E := E_1 + \cdots + E_r. \tag{2.4}$$

Then we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{g}(0) + \mathfrak{g}(1/2) + \mathfrak{g}(1)$ for the operator $\mathrm{ad}(H)$, where

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}, \qquad \mathfrak{g}(1/2) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i/2}, \qquad \mathfrak{g}(1) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m + \alpha_k)/2}.$$

Clearly, $[\,\mathfrak{g}(i), \mathfrak{g}(j)\,] \subset \mathfrak{g}(i + j)$ by understanding $\mathfrak{g}(k) = 0$ for $k > 1$. Moreover

$$J\mathfrak{n}_{(\alpha_m - \alpha_k)/2} = \mathfrak{n}_{(\alpha_m + \alpha_k)/2} \quad (m > k), \qquad J\mathfrak{n}_{\alpha_i/2} = \mathfrak{n}_{\alpha_i/2} \quad (1 \leqslant i \leqslant r), \tag{2.5}$$

so that $J\mathfrak{g}(0) = \mathfrak{g}(1)$ and $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$. We remark here that

$$JT = -[T, E] \quad \big(T \in \mathfrak{g}(0)\big), \qquad JT_{ji} = -[T_{ji}, E_i] \quad (T_{ji} \in \mathfrak{n}_{(\alpha_j - \alpha_i)/2}). \tag{2.6}$$

The following is a list of constants used in this paper:

$$\begin{aligned}
&n_{mk} := \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m-\alpha_k)/2} = \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m+\alpha_k)/2} && (1 \leqslant k < m \leqslant r),\\
&p_j := \sum_{k>j} n_{kj}, \qquad q_j := \sum_{i<j} n_{ji} && (1 \leqslant j \leqslant r),\\
&b_j := \tfrac{1}{2}\dim_{\mathbb{R}} \mathfrak{n}_{\alpha_j/2}, \qquad d_j := 1 + \tfrac{1}{2}(p_j+q_j) && (1 \leqslant j \leqslant r),\\
&\omega_k := \langle E_k, \omega\rangle = \|E_k\|_\omega^2 > 0 && (1 \leqslant k \leqslant r).
\end{aligned} \tag{2.7}$$

2.2. Homogeneous Siegel domains

Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra, and $G = \exp\mathfrak{g}$ the connected and simply connected Lie group corresponding to $\mathfrak{g}$. We denote by $G(0)$ the subgroup $\exp\mathfrak{g}(0)$ of G. By Section 2.1, we know that $G(0)$ acts on $V := \mathfrak{g}(1)$ by adjoint action. Recall $E \in V$ in (2.4) and let Ω be the $G(0)$-orbit through E. By [17, Theorem 4.15] Ω is a regular open convex cone in V, and $G(0)$ acts on Ω simply transitively. By (2.5) the subspace $\mathfrak{g}(1/2)$ is invariant under J, so that it is considered as a *complex* vector space U by means of $-J$. We put $W := V_{\mathbb{C}}$, the complexification of V. The conjugation of W relative to the real form V is written as $w \mapsto w^*$. The real bilinear map Q defined by

$$Q(u,u') := \tfrac{1}{2}\bigl([Ju,u'] - i[u,u']\bigr) \quad \bigl(u,u' \in \mathfrak{g}(1/2)\bigr) \tag{2.8}$$

turns out to be a sesqui-linear (complex linear in the first variable and antilinear in the second) Ω-positive Hermitian map $U \times U \to W$. We have

$$Q(u',u) = Q(u,u')^* \quad (u,u' \in U),$$

$$Q(u,u) \in \overline{\Omega} \setminus \{0\} \quad \text{for all } u \in U \setminus \{0\}.$$

The *Siegel domain* $D = D(\Omega, Q)$ defined by these data is

$$D := \bigl\{(u,w) \in U \times W;\ w + w^* - Q(u,u) \in \Omega\bigr\}. \tag{2.9}$$

Every homogeneous Siegel domain arises in this way.

Consider $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$. It is a nilpotent subalgebra of at most 2-step. Let $N_D = \exp\mathfrak{n}_D$ be the corresponding connected and simply connected nilpotent Lie group contained in G. Writing elements of N_D by $n(a,b)$ $(a \in \mathfrak{g}(1),\ b \in \mathfrak{g}(1/2))$, we see by the Campbell–Hausdorff formula that the group operation is described as

$$n(a,b)n(a',b') = n\bigl(a + a' - \operatorname{Im} Q(b,b'),\ b + b'\bigr). \tag{2.10}$$

The group N_D acts on D by

$$n(a,b)\cdot(u,w) = \bigl(u + b,\ w + ia + \tfrac{1}{2}Q(b,b) + Q(u,b)\bigr) \quad \bigl((u,w) \in D\bigr).$$

On the other hand, the adjoint action of $G(0)$ on $\mathfrak{g}(1/2)$ commutes with J. In other words, $G(0)$ acts on U complex-linearly. Moreover the adjoint action of $G(0)$ on $V = \mathfrak{g}(1)$ extends complex-linearly to W, so that $G(0)$ acts on D complex-linearly. In this way $G = N_D \rtimes G(0)$ acts on D simply transitively.

Let us put $A := \exp\mathfrak{a}$ and set for $t = (t_1, \dots, t_r) \in \mathbb{R}^r$

$$a_t := \exp(t_1 H_1 + \cdots + t_r H_r) \in A. \tag{2.11}$$

For every $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$, let $\chi_{\mathbf{s}}$ be the one-dimensional representation of A defined by $\chi_{\mathbf{s}}(a_t) = \exp(\sum_k s_k t_k)$. We put

$$\mathfrak{n}_0 := \sum_{m>k} \mathfrak{n}_{(\alpha_m-\alpha_k)/2}. \tag{2.12}$$

Clearly $\mathfrak{n}_0$ is a nilpotent Lie subalgebra of $\mathfrak{g}(0)$, and we have $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_D$. Let $N_0 := \exp \mathfrak{n}_0$ and $N := \exp \mathfrak{n}$. It is also clear that $G = N \rtimes A$ and $G(0) = N_0 \rtimes A$. We extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of G by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let us define functions $\Delta_{\mathbf{s}}$ ($\mathbf{s} \in \mathbb{R}^r$) on Ω by the transfer of $\chi_{\mathbf{s}}|_{G(0)}$:

$$\Delta_{\mathbf{s}}(hE) = \chi_{\mathbf{s}}(h) \quad \bigl(h \in G(0)\bigr). \tag{2.13}$$

Evidently it holds that

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x) \quad \bigl(h \in G(0),\ x \in \Omega\bigr). \tag{2.14}$$

In particular, putting $h = \exp tH \in A$ with $t = \log\lambda$ ($\lambda > 0$), we see that $\Delta_{\mathbf{s}}(\lambda x) = \lambda^{|\mathbf{s}|}\Delta_{\mathbf{s}}(x)$, where $|\mathbf{s}| := s_1 + \cdots + s_r$ for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$. Furthermore, we know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for example [7, Corollary 2.5]).

For $h \in G(0)$, let $\mathrm{Ad}_{\mathfrak{g}(1)}(h) := (\mathrm{Ad}\, h)|_{\mathfrak{g}(1)}$. Moreover let $\mathrm{Ad}_U(h)$ stand for the *complex* linear operator on U defined by the adjoint action of $h \in G(0)$ on $\mathfrak{g}(1/2)$, and $\det \mathrm{Ad}_U(h)$ its determinant as a complex linear operator. Then, with $\mathbf{d} := (d_1, \ldots, d_r)$ and $\mathbf{b} := (b_1, \ldots, b_r)$, we have for $h \in G(0)$

$$\det \mathrm{Ad}_{\mathfrak{g}(1)}(h) = \chi_{\mathbf{d}}(h), \qquad |\det \mathrm{Ad}_U(h)|^2 = \chi_{\mathbf{b}}(h). \tag{2.15}$$

3. Pseudoinverses

3.1. Admissible linear forms

Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. We keep to the notation in Section 2. Though the following lemma is probably known to the specialists, we include here its proof for the sake of readers' convenience.

Lemma 3.1. *Any admissible linear form ω' on $\mathfrak{g}$ vanishes on the root spaces $\mathfrak{n}_\alpha$ for $\alpha \neq \alpha_k$ ($k = 1, \ldots, r$).*

Proof. We extend ω' to a complex linear form on $\mathfrak{g}_{\mathbb{C}}$. Let $x \in \mathfrak{g}(0) + \mathfrak{g}(1/2)$ and $a \in \mathfrak{a}$. Since $[Ja, Jx] \in [\mathfrak{g}(1),\ \mathfrak{g}(1) + \mathfrak{g}(1/2)] = \{0\}$, the integrability condition (2.1) leads us to

$$[a + iJa,\ x + iJx] = [a, x] + i([Ja, x] + [a, Jx]) = [a, x] + iJ[a, x].$$

On the other hand, since ω' is admissible, we have

$$\langle [a + iJa,\ x + iJx],\ \omega' \rangle = -\langle Ja \mid x \rangle_{\omega'} + i\langle a \mid x \rangle_{\omega'} - i\langle x \mid a \rangle_{\omega'} + \langle x \mid Ja \rangle_{\omega'} = 0.$$

These two formulas imply

$$\langle [a, x], \omega' \rangle = 0, \qquad \langle J[a, x], \omega' \rangle = 0. \tag{3.1}$$

Taking $x \in \mathfrak{n}_{\alpha_j/2}$ or $x \in \mathfrak{n}_{(\alpha_m - \alpha_k)/2}$ ($m > k$) in the first equality of (3.1), we see that $\omega'|_{\mathfrak{n}_{\alpha_j/2}} = 0$ or $\omega'|_{\mathfrak{n}_{(\alpha_m - \alpha_k)/2}} = 0$ respectively. Taking $x \in \mathfrak{n}_{(\alpha_m - \alpha_k)/2}$ ($m > k$) in the second equality in (3.1), we obtain $\omega'|_{\mathfrak{n}_{(\alpha_m + \alpha_k)/2}} = 0$ by (2.5). $\square$

Lemma 3.2. *The subspace $\mathfrak{a}$ is still the orthogonal complement of $\mathfrak{n}$ with respect to $\langle \cdot \mid \cdot \rangle_{\omega'}$ for any admissible linear form ω'.*

Proof. Since $\mathfrak{a}$ is commutative, we see easily that

$$[J\mathfrak{n}, \mathfrak{a}] \subset \mathfrak{n}_0 + \mathfrak{g}(1/2) + J\mathfrak{n}_0,$$

where $\mathfrak{n}_0$ is as in (2.12). This implies $\langle \mathfrak{n} \mid \mathfrak{a} \rangle_{\omega'} = \{0\}$ by Lemma 3.1. □

Defining $E_1^*, \ldots, E_r^* \in \mathfrak{g}(1)^*$ by $\langle E_i, E_j^* \rangle = \delta_{ij}$ and $E_j^* = 0$ on $J\mathfrak{n}_0$, we set for every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$

$$E_{\mathbf{s}}^* := s_1 E_1^* + \cdots + s_r E_r^*. \tag{3.2}$$

We regard $E_{\mathbf{s}}^*$ as an element of $\mathfrak{g}^*$ by further defining $E_{\mathbf{s}}^* = 0$ on $\mathfrak{g}(0) + \mathfrak{g}(1/2)$.

Lemma 3.3. *$E_{\mathbf{s}}^*$ is admissible if and only if $s_j > 0$ for every $j = 1, \ldots, r$.*

This lemma is shown in [11, Lemma 5.1]. We shall denote by $\langle x \mid y \rangle_{\mathbf{s}}$ the inner product $\langle [Jx, y], E_{\mathbf{s}}^* \rangle$ on $\mathfrak{g}$. The corresponding norm $\| \cdot \|_{\mathbf{s}}$ is given as follows:

$$\begin{aligned} \|T + u + x\|_{\mathbf{s}}^2 = {} & \sum_{i=1}^r s_i t_i^2 + \sum_{k>j} s_k \omega_k^{-1} \|T_{kj}\|_\omega^2 + \sum_{k=1}^r s_k \omega_k^{-1} \|u_k\|_\omega^2 \\ & + \sum_{i=1}^r s_i x_i^2 + \sum_{k>j} s_k \omega_k^{-1} \|X_{kj}\|_\omega^2, \end{aligned} \tag{3.3}$$

where $T = \sum_i t_i H_i + \sum_{k>j} T_{kj}$, $u = \sum_k u_k$ and $x = \sum_i x_i E_i + \sum_{k>j} X_{kj}$ with $t_i, x_i \in \mathbb{R}$, $T_{kj} \in \mathfrak{n}_{(\alpha_k - \alpha_j)/2}$, $X_{kj} \in \mathfrak{n}_{(\alpha_k + \alpha_j)/2}$ and $u_k \in \mathfrak{n}_{\alpha_k/2}$. On the other hand, admissibility of a linear form on $\mathfrak{g}$ is clearly independent of its value on $\mathfrak{a} = \mathfrak{n}^\perp$. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, we write $\mathbf{s} > 0$ if $s_j > 0$ for all $j = 1, \ldots, r$. We thus arrive at the following proposition.

Proposition 3.4. *The set of admissible linear forms on $\mathfrak{g}$ coincides with*

$$\mathfrak{a}^* + \{E_{\mathbf{s}}^*;\ \mathbf{s} > 0\},$$

so that the elements $E_{\mathbf{s}}^*$ with $\mathbf{s} > 0$ represent the admissible linear forms on $\mathfrak{g}$.

Remark 3.5. Let $\beta \in \mathfrak{g}^*$ be the *Koszul form* given by

$$\langle x, \beta \rangle := \operatorname{tr}\bigl(\operatorname{ad}(Jx) - J \operatorname{ad}(x)\bigr) \quad (x \in \mathfrak{g}). \tag{3.4}$$

We know that β is admissible and that the inner product $\langle \cdot \mid \cdot \rangle_\beta$ is the real part of the Hermitian inner product on $\mathfrak{g}$ induced by the Bergman metric of D up to a positive number multiple (see [9, Théorème 1]). Moreover by [11, Lemma 5.2] we have $\beta|_{\mathfrak{g}(1)} = E_{2\mathbf{d}+\mathbf{b}}^*$.

3.2. Explosion of $\Delta_{-\mathbf{s}}$ on the boundary for $\mathbf{s} > 0$

The purpose of this subsection is to show that if $\mathbf{s} > 0$, then $\Delta_{-\mathbf{s}}(x) \to \infty$ as x tends to a point x_0 on the boundary $\partial\Omega$ of Ω. Recalling definition (2.12) of $\mathfrak{n}_0$, we put

$$\mathfrak{l}_i := \sum_{j>i} \mathfrak{n}_{(\alpha_j - \alpha_i)/2} \quad (i = 1, 2, \ldots, r-1). \tag{3.5}$$

Then, $\mathfrak{n}_0 = \sum_{i=1}^{r-1} \mathfrak{l}_i$. Using (2.3), we see easily that

$$[\mathfrak{l}_i, \mathfrak{l}_i] = \{0\}, \qquad [\mathfrak{l}_k, \mathfrak{l}_i] \subset \mathfrak{l}_i \quad (k > i). \tag{3.6}$$

With $L = (L_1, \cdots, L_{r-1}) \in \mathfrak{l}_1 \times \cdots \times \mathfrak{l}_{r-1}$, every element $n \in N_0$ is written as

$$n = n(L) := \exp(L_1)\exp(L_2)\cdots\exp(L_{r-1}). \tag{3.7}$$

We decompose every $L_i \in \mathfrak{l}_i$ as $L_i = \sum_{j>i} T_{ji}$, where $T_{ji} \in \mathfrak{n}_{(\alpha_j-\alpha_i)/2}$.

Lemma 3.6. $n(L)E_j = E_j + [L_j, E_j] + \frac{1}{2}[L_j, [L_j, E_j]]$ *for* $j = 1, 2, \ldots, r$.

Proof. Let j be fixed and we rewrite $n(L)$ as

$$\exp(L_j)\bigl[\exp(-L_j)\exp(L_1)\exp(L_j)\bigr]\cdots\bigl[\exp(-L_j)\exp(L_{j-1})\exp(L_j)\bigr] \\ \times \exp(L_{j+1})\cdots\exp(L_{r-1}).$$

Since (3.6) says $\exp(-L_j)\exp(L_i)\exp(L_j) \in \exp \mathfrak{l}_i$ for $1 \leqslant i \leqslant j-1$ and since (2.3) implies $[\mathfrak{l}_k, E_j] = \{0\}$ for $k \neq j$, we have $n(L)E_j = (\exp L_j)E_j$, so that

$$n(L)E_j = E_j + [L_j, E_j] + \frac{1}{2}\bigl[L_j, [L_j, E_j]\bigr] + \cdots.$$

Now $[L_j, [L_j, E_j]] \in \sum_{m \geqslant k > j} \mathfrak{n}_{(\alpha_m+\alpha_k)/2}$. Therefore $[L_j, [L_j, [L_j, E_j]]] = 0$ by (2.3), whence the proof of the lemma is completed. □

We recall that every element $h \in G(0)$ is written as $h = a_t n(L)$ with a_t as in (2.11).

Lemma 3.7. *If* $h = a_t n(L) \in G(0)$*, then one has*

$$\langle hE, E_{\mathbf{s}}^* \rangle = \sum_{j=1}^{r} \left(s_j e^{t_j} + \sum_{k>j} \omega_k^{-1} s_k e^{t_k} \|T_{kj}\|_\omega^2 \right).$$

Proof. Since $[L_j, E_j] \in \sum_{k>j} \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$, it holds that $\langle a_t[L_j, E_j], E_{\mathbf{s}}^* \rangle = 0$. Moreover

$$\bigl[L_j, [L_j, E_j]\bigr] \in \sum_{k>j} \bigl[T_{kj}, [T_{kj}, E_j]\bigr] + \sum_{m>k>j} \mathfrak{n}_{(\alpha_m+\alpha_k)/2}.$$

Here we see easily that $[T_{kj}, [T_{kj}, E_j]] = \omega_k^{-1}\|T_{kj}\|_\omega^2 E_k$ by using (2.6). Therefore the lemma follows from Lemma 3.6. □

Proposition 3.8. *If* $\mathbf{s} > 0$*, then* $\Delta_{-\mathbf{s}}$ *is exploding on* $\partial\Omega$*, that is, if* $x \in \Omega$ *tends to* $x_0 \in \partial\Omega$*, then* $\Delta_{-\mathbf{s}}(x) \to \infty$.

Proof. Suppose that the sequence $h_\nu = a_{t^{(\nu)}} n(L^{(\nu)}) \in G(0)$, where $t^{(\nu)} \in \mathbb{R}^r$ and $L^{(\nu)} = (L_1^{(\nu)}, \ldots, L_r^{(\nu)})$, is such that $h_\nu E \to x_0 \in \partial\Omega$ as $\nu \to \infty$. Then we have $\sup_\nu \langle h_\nu E, E_{\mathbf{s}}^* \rangle < \infty$. This together with Lemma 3.7 says that there exists a constant $M > 0$ such that $t_j^{(\nu)} \leqslant M$ for all $j = 1, \ldots, r$ and all ν. Let us show

$$\sum_{j=1}^{r} s_j t_j^{(\nu)} \to -\infty \quad (\nu \to \infty). \tag{3.8}$$

Suppose not. Then we can find a constant $c > 0$ and a subsequence $\{\nu_m\}$ such that $\sum_j s_j t_j^{(\nu_m)} \geqslant -c$. Hence for every l we have

$$-c \leqslant M\left(\sum_{j\neq l} s_j\right) + s_l t_l^{(\nu_m)} \leqslant M|\mathbf{s}| + s_l t_l^{(\nu_m)}.$$

This means that the sequence $\{t_l^{(\nu_m)}\}_m$ is bounded also from below for every l, so that Lemma 3.7 shows that the sequence $\{T_{kj}^{(\nu_m)}\}_m$ is bounded for every pair k, j $(k > j)$. Hence we may assume that h_{ν_m} converges to an element $h_0 \in G(0)$ by taking a further subsequence if necessary. Then $x_0 = \lim h_{\nu_m} E = h_0 E \in \Omega$, contradicting the fact that x_0 is on the boundary of the open set Ω. Consequently (3.8) is true, and we obtain

$$\Delta_{-\mathbf{s}}(h_\nu E) = \chi_{-\mathbf{s}}(h_\nu) = \exp -\left(\sum_j s_j t_j^{(\nu)}\right) \to \infty \quad (\nu \to \infty).$$

The proof is now complete. □

Let Ω^* be the dual cone of Ω defined by

$$\Omega^* := \left\{\xi \in V^*;\ \langle x, \xi\rangle > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\}\right\}.$$

The group $G(0)$ acts on Ω^* simply transitively by the coadjoint action: $h \cdot \xi = \xi \circ h^{-1}$, where $h \in G(0)$ and $\xi \in V^*$, and $\Omega^* = G(0) \cdot E_{\mathbf{1}}^*$ with $\mathbf{1} = (1, \ldots, 1)$, see [17]. We note that if $\mathbf{s} > 0$, then $E_{\mathbf{s}}^* \in \Omega^*$, because we have $E_{\mathbf{s}}^* = h_{\mathbf{s}} \cdot E_{\mathbf{1}}^* \in \Omega^*$, where

$$h_{\mathbf{s}} := \exp -\big((\log s_1) H_1 + \cdots + (\log s_r) H_r\big) \in A.$$

Choosing $E_{\mathbf{s}}^*$ ($\mathbf{s} > 0$) as a base point of Ω^*, we define a function $\Delta_{\mathbf{s}}^*$ on Ω^* by

$$\Delta_{\mathbf{s}}^*(h \cdot E_{\mathbf{s}}^*) := \chi_{\mathbf{s}}(h) \quad \big(h \in G(0)\big). \tag{3.9}$$

Clearly we have $\Delta_{\mathbf{s}}^*(h \cdot \xi) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}^*(\xi)$ for $h \in G(0)$ and $\xi \in \Omega^*$.

Proposition 3.9. *If* $\mathbf{s} > 0$, *then* $\Delta_{\mathbf{s}}^*$ *is exploding on* $\partial\Omega^*$.

Proof. This time we write every element $h \in G(0)$ as follows:

$$h = \exp(L_{r-1}) \cdots \exp(L_1) a_t. \tag{3.10}$$

Then we have by Lemma 3.7

$$\langle E, h \cdot E_{\mathbf{s}}^*\rangle = \big\langle h^{-1}E, E_{\mathbf{s}}^*\big\rangle = \sum_{j=1}^{r}\left(s_j e^{-t_j} + \sum_{k>j} \omega_k^{-1} s_k e^{-t_k} \|T_{kj}\|_\omega^2\right).$$

An argument similar to the proof of Proposition 3.8 leads us to the fact that if $h \cdot E_{\mathbf{s}}^*$ tends to a point $\xi \in \partial\Omega^*$ with $h \in G(0)$ as in (3.10), then $\sum_{j=1}^r s_j t_j \to \infty$. This implies

$$\Delta_{\mathbf{s}}^*(h \cdot E_{\mathbf{s}}^*) = \chi_{\mathbf{s}}(h) = \exp\left(\sum_j s_j t_j\right) \to \infty,$$

whence we get the proposition. □

3.3. Pseudoinverse maps

Let D_v be the directional derivative in the direction $v \in V$ given by

$$D_v f(x) = \frac{d}{dt} f(x + tv)\Big|_{t=0}.$$

Since (2.6) gives for $v \in V$

$$(\mathrm{Ad}\exp tJv)E = E + t[Jv, E] + \mathrm{O}\big(t^2\big) = E + tv + \mathrm{O}\big(t^2\big) \quad (t \in \mathbb{R}), \tag{3.11}$$

it holds that for smooth functions f on Ω

$$D_v f(E) = \frac{d}{dt} f\big((\mathrm{Ad}\exp tJv)E\big)\Big|_{t=0}. \tag{3.12}$$

Let $\mathbf{s} \in \mathbb{R}^r$ with $\mathbf{s} > 0$. For every $x \in \Omega$ we define $\mathcal{I}_{\mathbf{s}}(x) \in V^*$ by

$$\langle v, \mathcal{I}_{\mathbf{s}}(x)\rangle = -D_v \log \Delta_{-\mathbf{s}}(x) \quad (v \in V). \tag{3.13}$$

It is easy to show by using (2.14) that $\mathcal{I}_{\mathbf{s}}$ is $G(0)$-equivariant:

$$\mathcal{I}_{\mathbf{s}}(hx) = h \cdot \mathcal{I}_{\mathbf{s}}(x) \quad \big(h \in G(0),\ x \in \Omega\big). \tag{3.14}$$

In particular, $\mathcal{I}_{\mathbf{s}}(\lambda x) = \lambda^{-1}\mathcal{I}_{\mathbf{s}}(x)$ for all $\lambda > 0$.

Lemma 3.10.

(i) $D_v \Delta_{-\mathbf{s}}(E) = -\langle v, E_{\mathbf{s}}^*\rangle$ *for every* $v \in V$.
(ii) *One has* $\mathcal{I}_{\mathbf{s}}(E) = E_{\mathbf{s}}^*$.

The proof is completely parallel to that of [10, Lemma 2.1] and omitted.

Lemma 3.11. *The function* $\log \Delta_{-\mathbf{s}}$ *is strictly convex for* $\mathbf{s} > 0$.

Proof. For every $x \in \Omega$, we define a symmetric operator $H_{\mathbf{s}}(x)$ on V by

$$\langle H_{\mathbf{s}}(x)v_1 \mid v_2\rangle_{\mathbf{s}} = D_{v_1} D_{v_2} \log \Delta_{-\mathbf{s}}(x) \quad (v_1, v_2 \in V).$$

Our task is to show that $H_{\mathbf{s}}(x)$ is positive definite. An easy computation gives

$$H_{\mathbf{s}}(hx) = {}^{\mathrm{t}}\big(\mathrm{Ad}_V\, h^{-1}\big) H_{\mathbf{s}}(x) \big(\mathrm{Ad}_V\, h^{-1}\big) \quad \big(h \in G(0)\big),$$

where ${}^{\mathrm{t}}T$ denotes the transpose of the operator T on V with respect to the inner product $\langle \cdot \mid \cdot\rangle_{\mathbf{s}}$. Since $G(0)$ acts on Ω transitively, it is enough to show the positive-definiteness of $H_{\mathbf{s}}(E)$. We accomplish this by proving that $H_{\mathbf{s}}(E)$ is the identity operator, that is, by proving

$$D_{v_1} D_{v_2} \log \Delta_{-\mathbf{s}}(E) = \langle [Jv_1, v_2], E_{\mathbf{s}}^*\rangle \quad (v_1, v_2 \in V). \tag{3.15}$$

In fact by (3.12), (3.13) and (3.14) we have for $v \in V$

$$D_v^2 \log \Delta_{-\mathbf{s}}(E) = -\frac{d}{dt}\big\langle v, \mathcal{I}_{\mathbf{s}}\big((\mathrm{Ad}\exp tJv)E\big)\big\rangle\Big|_{t=0} = \langle (\mathrm{ad}\, Jv)v, \mathcal{I}_{\mathbf{s}}(E)\rangle.$$

Lemma (3.10)(ii) and polarization give (3.15). □

Proposition 3.12. *The map $\mathcal{I}_{\mathbf{s}} : x \mapsto \mathcal{I}_{\mathbf{s}}(x)$ gives a bijection of Ω onto Ω^*.*

Proof. Once we have the exploding property of $\Delta_{-\mathbf{s}}$ on $\partial\Omega$ and the strict convexity of $\log \Delta_{-\mathbf{s}}$, Proposition 3.12 follows through the standard argument as given in [18, §4] and [4, §4] (see also [5, §I.3]). We write down the proof here for completeness. Since $\mathcal{I}_{\mathbf{s}}(hE) = h \cdot \mathcal{I}_{\mathbf{s}}(E) = h \cdot E^*_{\mathbf{s}} \in \Omega^*$ for $h \in G(0)$, the map $\mathcal{I}_{\mathbf{s}}$ certainly sends Ω into Ω^*. By Taylor expansion, we have for $x \in \Omega$, $v \in V$ such that $x + v \in \Omega$, and $0 < \theta < 1$

$$\log \Delta_{-\mathbf{s}}(x + v) = \log \Delta_{-\mathbf{s}}(x) - \langle v, \mathcal{I}_{\mathbf{s}}(x)\rangle + \tfrac{1}{2}\langle H_{\mathbf{s}}(x + \theta v)v \mid v\rangle_{\mathbf{s}}.$$

Given distinct $a, b \in \Omega$, we put $x = a$ and $v = b - a \neq 0$. We obtain by Lemma 3.11

$$\log \frac{\Delta_{-\mathbf{s}}(a)}{\Delta_{-\mathbf{s}}(b)} - \langle b - a, \mathcal{I}_{\mathbf{s}}(a)\rangle = -\frac{1}{2}\langle H_{\mathbf{s}}(x + \theta v)v \mid v\rangle_{\mathbf{s}} < 0.$$

This implies

$$\frac{\Delta_{-\mathbf{s}}(a)}{\Delta_{-\mathbf{s}}(b)} < \exp\langle b - a, \mathcal{I}_{\mathbf{s}}(a)\rangle.$$

If we had $\mathcal{I}_{\mathbf{s}}(a) = \mathcal{I}_{\mathbf{s}}(b)$, then

$$\frac{\Delta_{-\mathbf{s}}(a)}{\Delta_{-\mathbf{s}}(b)} < \exp\langle b - a, \mathcal{I}_{\mathbf{s}}(a)\rangle = \exp -\langle a - b, \mathcal{I}_{\mathbf{s}}(b)\rangle < \frac{\Delta_{-\mathbf{s}}(a)}{\Delta_{-\mathbf{s}}(b)},$$

which would be a contradiction. Hence $\mathcal{I}_{\mathbf{s}}$ is injective.

To prove the surjectivity, we recall first that $\Delta_{-\mathbf{s}}(\lambda x) = \lambda^{-|\mathbf{s}|}\Delta_{-\mathbf{s}}(x)$ for $\lambda > 0$, so that we have

$$D_x \Delta_{-\mathbf{s}}(x) = \frac{d}{dt}\Delta_{-\mathbf{s}}(x + tx)\bigg|_{t=0} = -|\mathbf{s}|\Delta_{-\mathbf{s}}(x) \quad (x \in \Omega).$$

Hence we get

$$\langle x, \mathcal{I}_{\mathbf{s}}(x)\rangle = -D_x \log \Delta_{-\mathbf{s}}(x) = -\frac{D_x \Delta_{-\mathbf{s}}(x)}{\Delta_{-\mathbf{s}}(x)} = |\mathbf{s}|.$$

Now given $\xi \in \Omega^*$, we consider the hyperplane $\mathcal{H}_{\mathbf{s}}(\xi) := \{y \in V;\ \langle y, \xi\rangle = |\mathbf{s}|\}$ and $\mathcal{L} := \Omega \cap \mathcal{H}_{\mathbf{s}}(\xi)$. By [5, I.1.6], we know that the closure $\overline{\mathcal{L}}$ is compact. Since $\Delta_{-\mathbf{s}}$ is exploding on $\partial\Omega$ by Proposition 3.8, the minimum of $\log \Delta_{-\mathbf{s}}$ on $\overline{\mathcal{L}}$ is attained at a point $x_0 \in \mathcal{L}$. Then the method of Lagrange multipliers tells us that $\mathcal{I}_{\mathbf{s}}(x_0) = \lambda\xi$ for some $\lambda \in \mathbb{R}$. Therefore we obtain

$$|\mathbf{s}| = \langle x_0, \mathcal{I}_{\mathbf{s}}(x_0)\rangle = \lambda\langle x_0, \xi\rangle = \lambda|\mathbf{s}|,$$

whence $\lambda = 1$, so that $\mathcal{I}_{\mathbf{s}}(x_0) = \xi$. □

We call the map $\mathcal{I}_{\mathbf{s}}$ ($\mathbf{s} > 0$) the *pseudoinverse map*. For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ we put

$$\alpha_{\mathbf{s}} := s_1\alpha_1 + \cdots + s_r\alpha_r \in \mathfrak{a}^*.$$

We extend $\alpha_{\mathbf{s}}$ to a Lie algebra representation of $\mathfrak{g}(0)$ by setting $\alpha_{\mathbf{s}}|_{\mathfrak{n}_0} = 0$. We have $\chi_{\mathbf{s}}(\exp T) = e^{\alpha_{\mathbf{s}}(T)}$ for $T \in \mathfrak{g}(0)$ and $\langle Jv, \alpha_{\mathbf{s}}\rangle = \langle v, E^*_{\mathbf{s}}\rangle$ for $v \in V$. We write down here the explicit dependence of $\Delta_{-\mathbf{s}}$ (hence of $\mathcal{I}_{\mathbf{s}}$) for $\mathbf{s} > 0$ on the admissible linear form $E^*_{\mathbf{s}}$ to clarify the matters:

$$\Delta_{-\mathbf{s}}(hE) = \chi_{-\mathbf{s}}(h) = e^{-\langle \log h, \alpha_{\mathbf{s}}\rangle} = e^{\langle J(\log h), E^*_{\mathbf{s}}\rangle} \quad \big(h \in G(0)\big), \tag{3.16}$$

where log is the inverse map of the diffeomorphism $\exp : \mathfrak{g}(0) \to G(0)$.

3.4. Dual pseudoinverse maps

We continue to assume $\mathbf{s} > 0$. For every $f \in V^*$, we define $\iota_{\mathbf{s}}(f) \in V$ by

$$\langle v, f\rangle = \langle v \mid \iota_{\mathbf{s}}(f)\rangle_{\mathbf{s}} \quad (\text{for all } v \in V). \tag{3.17}$$

It is evident that $\iota_{\mathbf{s}}$ is a linear bijection from V^* onto V. The formula (3.3) for $T = 0$ and $u = 0$ shows

$$\iota_{\mathbf{s}}(E_{\mathbf{s}}^*) = E. \tag{3.18}$$

For $\xi \in \Omega^*$ we define $\mathcal{I}_{\mathbf{s}}^*(\xi) \in V$ by

$$\langle \mathcal{I}_{\mathbf{s}}^*(\xi), f\rangle := -D_f \log \Delta_{\mathbf{s}}^*(\xi) \quad (f \in V^*).$$

It is easy to see that $\mathcal{I}_{\mathbf{s}}^*$ is $G(0)$-equivariant, that is,

$$\mathcal{I}_{\mathbf{s}}^*(h\cdot\xi) = h\big(\mathcal{I}_{\mathbf{s}}^*(\xi)\big) \quad \big(h \in G(0)\big).$$

Lemma 3.13. *For every $f \in V^*$, one has*

$$D_f \Delta_{\mathbf{s}}^*(E_{\mathbf{s}}^*) = -\langle E, f\rangle, \qquad \mathcal{I}_{\mathbf{s}}^*(E_{\mathbf{s}}^*) = E.$$

Proof. Given $f \in V^*$, we consider the element $T := J\iota_{\mathbf{s}}(f) \in \mathfrak{g}(0)$. Then (3.17) gives for $v \in V$

$$\begin{aligned}\langle v, \mathrm{Ad}^*(\exp -tT)E_{\mathbf{s}}^*\rangle &= \langle v, E_{\mathbf{s}}^*\rangle + t\,\langle [J\iota_{\mathbf{s}}(f), v], E_{\mathbf{s}}^*\rangle + \mathrm{O}\big(t^2\big)\\ &= \langle v, E_{\mathbf{s}}^*\rangle + t\,\langle v, f\rangle + \mathrm{O}\big(t^2\big).\end{aligned} \tag{3.19}$$

Therefore we get

$$D_f \Delta_{\mathbf{s}}^*(E_{\mathbf{s}}^*) = \frac{d}{dt}\, \Delta_{\mathbf{s}}^*\big(\mathrm{Ad}^*(\exp -tT)E_{\mathbf{s}}^*\big)\bigg|_{t=0} = -\langle T, \alpha_{\mathbf{s}}\rangle = -\langle \iota_{\mathbf{s}}(f), E_{\mathbf{s}}^*\rangle.$$

Since $\langle \iota_{\mathbf{s}}(f), E_{\mathbf{s}}^*\rangle = \langle \iota_{\mathbf{s}}(f) \mid E\rangle_{\mathbf{s}} = \langle E, f\rangle$ by (3.18), the first assertion follows. The second assertion follows from

$$\langle \mathcal{I}_{\mathbf{s}}^*(E_{\mathbf{s}}^*), f\rangle = -\frac{D_f \Delta_{\mathbf{s}}^*(E_{\mathbf{s}}^*)}{\Delta_{\mathbf{s}}^*(E_{\mathbf{s}}^*)} = \langle E, f\rangle.$$

The proof is now complete. □

Lemma 3.14. *The function* $\log \Delta_{\mathbf{s}}^*$ *is strictly convex for* $\mathbf{s} > 0$.

Proof. For each $\xi \in \Omega^*$, let us define a symmetric operator $H_{\mathbf{s}}^*(\xi)$ on V by

$$D_{f_1} D_{f_2} \log \Delta_{\mathbf{s}}^*(\xi) = \langle H_{\mathbf{s}}^*(\xi)\iota_{\mathbf{s}}(f_1) \mid \iota_{\mathbf{s}}(f_2)\rangle_{\mathbf{s}} \quad (f_1, f_2 \in V^*).$$

Then an easy computation shows

$$H_{\mathbf{s}}^*(h\cdot\xi) = (\mathrm{Ad}_V\, h)\, H_{\mathbf{s}}^*(\xi)\,{}^{\mathrm{t}}(\mathrm{Ad}_V\ h) \quad \big(h \in G(0)\big).$$

Thus it is sufficient to show that $H_{\mathbf{s}}^*(E_{\mathbf{s}}^*)$ is the identity operator. This can be shown by using (3.19) just as in the proof of Lemma 3.11. Details are left to the reader. □

The preceding lemma together with Proposition 3.9 yields

Proposition 3.15. *The map $\mathcal{I}^*_{\mathbf{s}} : \xi \mapsto \mathcal{I}^*_{\mathbf{s}}(\xi)$ gives rise to a bijection of Ω^* onto Ω.*

The proof is given by a discussion parallel to Proposition 3.12, so omitted. We call the map $\mathcal{I}^*_{\mathbf{s}}$ ($\mathbf{s} > 0$) the *dual pseudoinverse map*.

Proposition 3.16. $\mathcal{I}^*_{\mathbf{s}}(\mathcal{I}_{\mathbf{s}}(x)) = x$ *for any* $x \in \Omega$, *and* $\mathcal{I}_{\mathbf{s}}(\mathcal{I}^*_{\mathbf{s}}(\xi)) = \xi$ *for any* $\xi \in \Omega^*$.

Proof. This is a direct consequence of the formulas $\mathcal{I}_{\mathbf{s}}(E) = E^*_{\mathbf{s}}$ of Lemma 3.10 and $\mathcal{I}^*_{\mathbf{s}}(E^*_{\mathbf{s}}) = E$ of Lemma 3.13 together with the fact that both $\mathcal{I}_{\mathbf{s}}$ and $\mathcal{I}^*_{\mathbf{s}}$ are $G(0)$-equivariant. □

3.5. Birationality of $\mathcal{I}_{\mathbf{s}}$

To see first that $\mathcal{I}_{\mathbf{s}}$ is a rational map, we proceed as in [3, Satz 3.3] and [10, 2.2]. We introduce a (non-associative) product $\star$ in V by

$$v_1 \star v_2 := [Jv_1, v_2] = \big(\mathrm{ad}(Jv_1)\big)v_2 \quad (v_1, v_2 \in V). \tag{3.20}$$

Note that by (2.6), the map $v \mapsto Jv$ is just an inverse map to the linear isomorphism $\mathfrak{g}(0) \ni T \to [T, E] \in \mathfrak{g}(1)$, which is the differential of the orbit map $G(0) \ni h \mapsto hE \in \Omega$. We shall write $R_J(v_2)v_1 = v_1 \star v_2$. Then, $R_J(E)v = [Jv, E] = v$ by (2.6), so that $R_J(E)$ is the identity operator. The product $\star$ extends to $W = V_{\mathbb{C}}$ by complex bilinearity, where J is also continued to a complex linear operator. We still denote by $R_J(w)$ the right multiplication by $w \in W$ on W. Then $w \mapsto \det R_J(w)$ is a non-zero polynomial function on W. Let us consider $\mathcal{O} := \{w \in W;\ \det R_J(w) \neq 0\}$. Clearly $\mathcal{O}$ is a non-empty Zariski-open set.

Lemma 3.17. *The pseudoinverse map $\mathcal{I}_{\mathbf{s}}$ ($\mathbf{s} > 0$) can be continued analytically to a rational map $W \to W^*$, and one has $\mathcal{I}_{\mathbf{s}}(w) = E^*_{\mathbf{s}} \circ R_J(w)^{-1}$ for $w \in \mathcal{O}$.*

Proof. The map $w \mapsto E^*_{\mathbf{s}} \circ R_J(w)^{-1}$ is clearly a rational map $\mathcal{O} \to W^*$ which coincides with $\mathcal{I}_{\mathbf{s}}$ on Ω just as in the proof of [10, Lemma 2.4] with η replaced by $\Delta_{-\mathbf{s}}$. We would like to point out here that in that proof v_2 should be in Ω. □

Similarly, the dual pseudoinverse map $\mathcal{I}^*_{\mathbf{s}}$ ($\mathbf{s} > 0$) can be continued analytically to a rational map $W^* \to W$. Therefore Proposition 3.16 yields

Theorem 3.18. *Suppose* $\mathbf{s} > 0$. *The rational map $\mathcal{I}^*_{\mathbf{s}}$ is inverse to the rational map $\mathcal{I}_{\mathbf{s}}$. In particular, $\mathcal{I}_{\mathbf{s}}$ is birational.*

Theorem 3.19. *Suppose* $\mathbf{s} > 0$, *so that $E^*_{\mathbf{s}}$ is admissible.*

(i) *$\mathcal{I}_{\mathbf{s}}$ is holomorphic on $\Omega + iV$, and $\mathcal{I}^*_{\mathbf{s}}$ is holomorphic on $\Omega^* + iV^*$.*
(ii) *$\mathcal{I}_{\mathbf{s}}(\Omega + iV)$ is contained in the holomorphic domain of $\mathcal{I}^*_{\mathbf{s}}$, and $\mathcal{I}^*_{\mathbf{s}}(\Omega^* + iV^*)$ in the holomorphic domain of $\mathcal{I}_{\mathbf{s}}$.*

Theorem 3.19 is proved by showing

$$\Omega + iV \subset G(0)(E + iV) \subset G(0)_{\mathbb{C}}E \subset \mathcal{O},$$

where $G(0)_{\mathbb{C}}$ is the complexification of the split solvable Lie group $G(0)$, and the second inclusion is a consequence of [10, Proposition 2.8]. The details are completely parallel to the proof of Theorems 2.10 and 2.11 in [10]. We do not repeat them here.

4. Cayley transforms

4.1. Definition of Cayley transforms

We continue to suppose $\mathbf{s} > 0$, so that $E_{\mathbf{s}}^*$ is an admissible linear form. Considering $E_{\mathbf{s}}^*$ canonically as an element of W^*, we now define for $w \in W$

$$C_{\mathbf{s}}(w) := E_{\mathbf{s}}^* - 2\mathcal{I}_{\mathbf{s}}(w+E). \tag{4.1}$$

It is evident that $C_{\mathbf{s}}$ is a rational mapping $W \to W^*$ which is holomorphic on $\Omega + iV$. Let $U^\dagger$ denote the space of all antilinear forms on U. We set for $z = (u, w) \in U \times W$

$$\mathcal{C}_{\mathbf{s}}(z) := \bigl(2\langle Q(u,\cdot), \mathcal{I}_{\mathbf{s}}(w+E)\rangle,\ C_{\mathbf{s}}(w)\bigr). \tag{4.2}$$

Clearly $\mathcal{C}_{\mathbf{s}}$ is a rational map $U \times W \to U^\dagger \times W^*$. It should be noted that if $z = (u, w) \in D$, then we have $w \in \Omega + iV$, so that $\mathcal{C}_{\mathbf{s}}(z)$ is holomorphic on D. We shall call $\mathcal{C}_{\mathbf{s}}$ the **s**-*Cayley transform*. The Cayley transform treated in [10] is the $(2\mathbf{d}+\mathbf{b})$-Cayley transform, and Penney's in [14] is the **d**-Cayley transform in the current terminology. We need the $(\mathbf{d}+\mathbf{b})$-Cayley transform in a forthcoming paper [13]. In [10] we have shown that the image $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(D)$ is bounded following the argument of Penney [14]. We remark that for general $\mathbf{s} > 0$, the image $\mathcal{C}_{\mathbf{s}}(D)$ for symmetric D is *not* the standard Harish-Chandra realization of a Hermitian symmetric space. This means that the previous way of proof does not work smoothly. In this paper, we show the boundedness of $\mathcal{C}_{\mathbf{s}}(D)$ in a different and straightforward way, which does not require the validity of the statement for symmetric domains separately.

4.2. Estimates for adjoint actions

We assume that the rank r of $\mathfrak{g}$ is greater than 1 until the end of the proof of Proposition 4.4. Recalling (3.5), we put $\mathfrak{l} := \mathfrak{l}_1$ and $\mathfrak{m} := J\mathfrak{l}$ for simplicity. We also put

$$\mathfrak{g}'(0) := \sum_{j=2}^{r} \mathbb{R}H_j \oplus \sum_{k>j\geqslant 2} \mathfrak{n}_{(\alpha_k-\alpha_j)/2}, \qquad V' := J\mathfrak{g}'(0).$$

We have $V = \mathbb{R}E_1 + \mathfrak{m} + V'$, so that every element $x \in V$ may be written as $x = x_1E_1 + X + x'$ with $x_1 \in \mathbb{R}$, $X \in \mathfrak{m}$ and $x' \in V'$. Let $G'(0) := \exp \mathfrak{g}'(0)$. By (2.3) the adjoint action of $G'(0)$ leaves both $\mathfrak{l}$ and $\mathfrak{m}$ invariant. Let $\mathrm{Ad}_{\mathfrak{l}}\, h$ and $\mathrm{Ad}_{\mathfrak{m}}\, h$ denote the corresponding restrictions for $h \in G'(0)$.

Lemma 4.1. *If* $h \in G'(0)$ *and* $Z \in \mathfrak{m}$, *then one has* $(\mathrm{Ad}_{\mathfrak{l}}\, h)JZ = J(\mathrm{Ad}_{\mathfrak{m}}\, h)Z$.

Proof. For brevity, let us write hZ and hJZ instead of $(\mathrm{Ad}_{\mathfrak{m}}\, h)Z$ and $(\mathrm{Ad}_{\mathfrak{l}}\, h)JZ$ respectively. By (2.6) we have $[JhZ, E_1] = hZ$. Since $hE_1 = E_1$, we get

$$Z = h^{-1}[JhZ, E_1] = \bigl[h^{-1}JhZ, E_1\bigr] = -Jh^{-1}JhZ,$$

where the last equality is again a consequence of (2.6). This clearly implies that $hJZ = JhZ$. □

Put $E' := E_2 + \cdots + E_r$. Then the $G'(0)$-orbit $\Omega' := G'(0)E'$ is a regular open convex cone in V'.

Lemma 4.2. *Suppose $x = x_1 E_1 + X + x' \in \overline{\Omega}$. Then $x_1 \geqslant 0$. If $x_1 > 0$, one has*

$$\bigl(\exp(-x_1^{-1} JX)\bigr)x = x_1 E_1 + x' - \tfrac{1}{2} x_1^{-1}[JX, X]. \tag{4.3}$$

In particular, $x' - \frac{1}{2} x_1^{-1}[JX, X] \in \overline{\Omega'}$.

Proof. Since $E_1^* \in \overline{\Omega^*}$, it is obvious that $x_1 = \langle x, E_1^* \rangle \geqslant 0$. Suppose now that $x_1 > 0$, and put $L := -x_1^{-1} JX \in \mathfrak{l}$. Then Lemma 3.6 and (2.6) say

$$\begin{aligned}(\exp L)E_1 &= E_1 + [L, E_1] + \tfrac{1}{2}[L, [L, E_1]] = E_1 - JL + \tfrac{1}{2}[JL, L] \\ &= E_1 - x_1^{-1} X + \tfrac{1}{2} x_1^{-2}[JX, X].\end{aligned}$$

Similarly $[L, [L, X]] = 0$ implies $(\exp L)X = X - x_1^{-1}[JX, X]$. Since $(\exp L)x' = x'$, we get (4.3). Since $(\exp(-x_1^{-1} JX))x$ is still in $\overline{\Omega}$ and since $x' - \frac{1}{2} x_1^{-1}[JX, X] \in V'$, the last assertion is clear. □

Corollary 4.3. *If $x = x_1 E_1 + X + x' \in \overline{\Omega}$, then $x' - \frac{1}{2}(x_1 + a)^{-1}[JX, X] \in \overline{\Omega'}$ for any $a > 0$.*

Proof. Just consider $x + aE_1 \in \overline{\Omega}$ in Lemma 4.2. □

Proposition 4.4. *If $Z \in \mathfrak{m}$, then one has $[JZ, Z] \in \overline{\Omega'}$.*

Proof. Let $\xi' \in (\Omega')^*$ be arbitrary. Put $F^* := E_2^* + \cdots + E_r^*$. Then there exists $h \in G'(0)$ such that $\xi' = h \cdot F^*$. We have

$$\langle [JZ, Z], \xi' \rangle = \bigl\langle h^{-1}[JZ, Z], F^* \bigr\rangle = \bigl\langle \bigl[Jh^{-1}Z, h^{-1}Z\bigr], F^* \bigr\rangle$$

by virtue of Lemma 4.1. Thus it suffices to show that $\langle [JY, Y], F^* \rangle \geqslant 0$ for any $Y \in \mathfrak{m}$. But if $Y = \sum_k Y_k$ with $Y_k \in \mathfrak{n}_{(\alpha_k + \alpha_1)/2}$, then we have

$$[JY, Y] \in \sum_k [JY_k, Y_k] + \sum_{2 \leqslant k < m} \mathfrak{n}_{(\alpha_m + \alpha_k)/2}.$$

Since $[JY_k, Y_k] = s_k^{-1} \|Y_k\|_{\mathbf{s}}^2 E_k$, we obtain $\langle [JY, Y], F^* \rangle = \sum_k s_k^{-1} \|Y_k\|_{\mathbf{s}}^2 \geqslant 0$. □

For every $x \in \overline{\Omega}$, let $h_x \in G(0)$ be the unique element for which $h_x E = E + x$. The following is the main result of this subsection.

Proposition 4.5. *There exists a positive constant $K_{\mathbf{s}}$ such that $\| \mathrm{Ad}_V\, h_x^{-1} \|_{\mathbf{s}} \leqslant K_{\mathbf{s}}$ for all $x \in \overline{\Omega}$.*

Proof. We prove the proposition by induction on the rank r of $\mathfrak{g}$. If $r = 1$, then $\mathrm{Ad}_V\, h_x$ $(x \geqslant 0)$ is the multiplication by $x + 1$, so that the proposition is true. Now we suppose that the proposition is true for normal j-algebras of rank $r - 1$. For every $y \in \overline{\Omega'}$, we denote by $h'_y \in G'(0)$ the unique element such that $h'_y E' = E' + y$. Induction hypothesis says that there is a positive constant $K'_{\mathbf{s}}$ such that

$$\left\| \mathrm{Ad}_{V'}(h'_y)^{-1} \right\|_{\mathbf{s}} \leqslant K'_{\mathbf{s}} \quad \text{for all } y \in \overline{\Omega'}. \tag{4.4}$$

Corollary 4.3 implies that if $x = x_1 E_1 + X + x' \in \overline{\Omega}$, then there is $h_0 \in G'(0)$ such that $h_0 E' = E' + x' - \frac{1}{2}(x_1 + 1)^{-1}[JX, X]$. Then by Lemma 4.2 we have

$$\left(\exp -(x_1 + 1)^{-1} JX\right)(x + E) = \left(\exp \log(x_1 + 1) H_1\right) h_0 E,$$

so that we obtain

$$h_x = \left(\exp(x_1 + 1)^{-1} JX\right)\left(\exp \log(x_1 + 1) H_1\right) h_0.$$

Here we have by (4.4)

$$\left\| \mathrm{Ad}_{V'} h_0^{-1} \right\|_{\mathbf{s}} \leqslant K'_{\mathbf{s}} \quad \text{for all } x \in \overline{\Omega}. \tag{4.5}$$

Let $a \in \mathbb{R}$, $Z \in \mathfrak{m}$ and $v' \in V'$. Then computing as in the proof of Lemma 4.2, we obtain

$$\begin{aligned}
&\left(\exp -(x_1 + 1)^{-1} JX\right)(aE_1 + Z + v') \\
&\quad = aE_1 + \left(Z - \frac{a}{x_1 + 1} X\right) + \left(v' - \frac{1}{x_1 + 1}[JX, Z] + \frac{a}{2(x_1 + 1)^2}[JX, X]\right).
\end{aligned}$$

Since $\exp(tH_1)(E_1 + Z + v') = e^t E_1 + e^{t/2} Z + v'$ $(t \in \mathbb{R})$, and since h_0 acts trivially on E_1, we arrive at

$$\begin{aligned}
h_x^{-1}(aE_1 + Z + v') = {}& \frac{a}{x_1 + 1} E_1 + \frac{1}{\sqrt{x_1 + 1}} \left(\mathrm{Ad}_{\mathfrak{m}}\, h_0^{-1}\right)\left(Z - \frac{a}{x_1 + 1} X\right) \\
& + \left(\mathrm{Ad}_{V'} h_0^{-1}\right)\left(v' - \frac{1}{x_1 + 1}[JX, Z] + \frac{a}{2(x_1 + 1)^2}[JX, X]\right).
\end{aligned} \tag{4.6}$$

By (4.5) and Lemma 4.1 we have only to estimate $\mathrm{Ad}_{\mathfrak{m}} h_0^{-1}$, $h_0^{-1} X$ and $h_0^{-1}[JX, X]$. Before proceeding further, we take here a positive constant $M_{\mathbf{s}}$ for which we have

$$\|[x, y]\|_{\mathbf{s}} \leqslant M_{\mathbf{s}} \|x\|_{\mathbf{s}} \|y\|_{\mathbf{s}} \quad \text{for all } x, y \in \mathfrak{g}. \tag{4.7}$$

Lemma 4.6. *One has* $\|\mathrm{Ad}_{\mathfrak{m}}\, g\|_{\mathbf{s}}^2 \leqslant M_{\mathbf{s}} \|E_{\mathbf{s}}^*\|_{\mathbf{s}} \|\mathrm{Ad}_{V'}\, g\|_{\mathbf{s}}$ *for any* $g \in G'(0)$.

Proof. Clearly we may assume $\mathfrak{m} \neq \{0\}$. Fix $g \in G'(0)$ and let $Z \in \mathfrak{m}$ be a unit eigenvector of the symmetric operator ${}^{t}(\mathrm{Ad}_{\mathfrak{m}}\, g)(\mathrm{Ad}_{\mathfrak{m}}\, g)$ corresponding to the maximum eigenvalue. Then Lemma 4.1 yields

$$\begin{aligned}
\|\mathrm{Ad}_{\mathfrak{m}}\, g\|_{\mathbf{s}}^2 &= \left\| {}^{t}(\mathrm{Ad}_{\mathfrak{m}}\, g)(\mathrm{Ad}_{\mathfrak{m}}\, g) \right\|_{\mathbf{s}} = \|gZ\|_{\mathbf{s}}^2 = \langle [JgZ, gZ], E_{\mathbf{s}}^* \rangle \\
&= \langle g[JZ, Z], E_{\mathbf{s}}^* \rangle \leqslant \|\mathrm{Ad}_{V'}\, g\|_{\mathbf{s}} \|[JZ, Z]\|_{\mathbf{s}} \|E_{\mathbf{s}}^*\|_{\mathbf{s}} \leqslant M_{\mathbf{s}} \|E_{\mathbf{s}}^*\|_{\mathbf{s}} \|\mathrm{Ad}_{V'}\, g\|_{\mathbf{s}}.
\end{aligned}$$

Hence we get the lemma. □

Lemma 4.7. $\|\mathrm{Ad}_{\mathfrak{m}}\, h_0^{-1}\|_{\mathbf{s}}$ *is bounded by a positive constant independent of* $x \in \overline{\Omega}$.

Proof. Immediate from Lemma 4.6 and (4.5). □

Lemma 4.8. *There is a positive constant* $N_{\mathbf{s}}$ *such that* $\|h_0^{-1}[JX, X]\|_{\mathbf{s}} \leqslant N_{\mathbf{s}}(x_1+1)^2$ *for all* $x \in \overline{\Omega}$.

Proof. We note that $x' \in \overline{\Omega'}$ and $x' - \frac{1}{2}(x_1+1/2)^{-1}[JX, X] \in \overline{\Omega'}$. We thus take $h_1, h_2 \in G'(0)$ such that

$$h_1 E' = E' + \tfrac{1}{2}(x_1+1)^{-1}x', \qquad h_2 E' = E' + \frac{x_1+\frac{1}{2}}{x_1+1} h_1^{-1}\bigl(x' - \tfrac{1}{2}\bigl(x_1+\tfrac{1}{2}\bigr)^{-1}[JX, X]\bigr).$$

We have $h_1 h_2 E' = E' + x' - \frac{1}{2}(x_1+1)^{-1}[JX, X] = h_0 E'$, so that it holds that $h_0 = h_1 h_2$. Since $\frac{1}{2}(x_1+1)^{-1}h_1^{-1}x' = E' - h_1^{-1}E'$, we get $\|h_1^{-1}x'\|_{\mathbf{s}} \leqslant 2N'_{\mathbf{s}}(x_1+1)$ for all $x \in \overline{\Omega}$ with $N'_{\mathbf{s}} := (1+K'_{\mathbf{s}})\|E'\|_{\mathbf{s}}$, where $K'_{\mathbf{s}}$ is the positive constant in (4.4). Applying (4.4) to h_2, we see that $\|h_0^{-1}x'\|_{\mathbf{s}} = \|h_2^{-1}h_1^{-1}x'\|_{\mathbf{s}} \leqslant 2K'_{\mathbf{s}}N'_{\mathbf{s}}(x_1+1)$. On the other hand, by the equality

$$h_0^{-1}\bigl(x' - \tfrac{1}{2}(x_1+1)^{-1}[JX, X]\bigr) = E' - h_0^{-1}E',$$

it holds that with the constant $N'_{\mathbf{s}}$ above

$$\bigl\|h_0^{-1}\bigl(x' - \tfrac{1}{2}(x_1+1)^{-1}[JX, X]\bigr)\bigr\|_{\mathbf{s}} \leqslant N'_{\mathbf{s}}.$$

Therefore we arrive at

$$\frac{1}{2(x_1+1)}\bigl\|h_0^{-1}[JX, X]\bigr\|_{\mathbf{s}} \leqslant N'_{\mathbf{s}}(1+2K'_{\mathbf{s}})(x_1+1),$$

from which the lemma follows immediately. □

Lemma 4.9. *With the constant* $N_{\mathbf{s}}$ *in Lemma* 4.8*, one has*

$$\bigl\|h_0^{-1}X\bigr\|_{\mathbf{s}} \leqslant N_{\mathbf{s}}^{1/2}\|E_{\mathbf{s}}^*\|_{\mathbf{s}}^{1/2}(x_1+1) \quad \textit{for all } x \in \overline{\Omega}.$$

Proof. Lemma 4.1 gives us

$$\bigl\|h_0^{-1}X\bigr\|_{\mathbf{s}}^2 = \bigl\langle\bigl[Jh_0^{-1}X, h_0^{-1}X\bigr], E_{\mathbf{s}}^*\bigr\rangle = \bigl\langle h_0^{-1}[JX, X], E_{\mathbf{s}}^*\bigr\rangle \leqslant \bigl\|h_0^{-1}[JX, X]\bigr\|_{\mathbf{s}}\|E_{\mathbf{s}}^*\|_{\mathbf{s}}.$$

Hence the lemma follows from Lemma 4.8. □

Now we conclude the proof of Proposition 4.5 by finishing the induction process using (4.6) and Lemmas 4.7, 4.8 and 4.9.

4.3. Estimate for $\mathcal{I}_{\mathbf{s}}$

The inner product $\langle\cdot \mid \cdot\rangle_{\mathbf{s}}$ on V extends to a complex bilinear form on $W \times W$, which we denote by the same symbol. Hence we have a Hermitian inner product $(w_1 \mid w_2)_{\mathbf{s}} := \langle w_1 \mid w_2^*\rangle_{\mathbf{s}}$ on W. The corresponding norm on W (and on W^*) will be expressed as $\|\cdot\|_{\mathbf{s}}$. The aim of this subsection is to prove the following theorem.

Theorem 4.10. *There exists a positive constant* $L_{\mathbf{s}}$ *such that*

$$\|\mathcal{I}_{\mathbf{s}}(x+E+iy)\|_{\mathbf{s}} \leqslant L_{\mathbf{s}} \quad \textit{for all } x \in \overline{\Omega} \textit{ and } y \in V.$$

In order to prove Theorem 4.10, we introduce the complexification $G_{\mathbb{C}}$ of G. Let $G(0)_{\mathbb{C}}$ be the complex analytic subgroup of $G_{\mathbb{C}}$ corresponding to the subalgebra $\mathfrak{g}(0)_{\mathbb{C}}$. Owing to the proof of [10, Proposition 2.8], there is a unique real analytic map $\eta : V \to G(0)_{\mathbb{C}}$ such that $\eta(y)E = E + iy$ and $\eta(0) = e$, the identity element of $G(0)_{\mathbb{C}}$. The principal step here is to estimate the action of $\eta(y)^{-1}$ on W. Although we proceed as in the previous subsection, we remark here that the proof of Lemma 4.6 does not work for getting an estimate for the action of $G'(0)_{\mathbb{C}}$ on $\mathfrak{m}_{\mathbb{C}}$. This makes the matters a little complicated to get an analogue of Lemma 4.7 (Corollary 4.15).

For every $a = 1, 2, \ldots, r$, we set

$$\mathfrak{g}_a(0) := \sum_{j=a}^{r} \mathbb{R}H_j \oplus \sum_{k>j\geqslant a} \mathfrak{n}_{(\alpha_k-\alpha_j)/2}, \qquad V_a := J\mathfrak{g}_a(0).$$

If $1 \leqslant a \leqslant r-1$, we have $V_a = \mathbb{R}E_a + \mathfrak{m}_a + V_{a+1}$ with $\mathfrak{m}_a := J\mathfrak{l}_a$. Let $G_a(0) := \exp \mathfrak{g}_a(0)$ and $G_a(0)_{\mathbb{C}}$ be its complexification. We put $F_a := E_a + \cdots + E_r$. Then we have the corresponding real analytic map $\eta_a : V_a \to G_a(0)_{\mathbb{C}}$ such that $\eta_a(y)F_a = F_a + iy$ with $\eta_a(0) = e$.

Lemma 4.11. *Suppose $1 \leqslant a < r$ and express $y \in V_a$ as $y = y_aE_a + Y + y'$ with $y_a \in \mathbb{R}$, $Y \in \mathfrak{m}_a$ and $y' \in V_{a+1}$. Then*

$$\left(\exp -\frac{iJY}{1+iy_a}\right)(F_a + iy) = (1+iy_a)E_a + F_{a+1} + iy' + \frac{1}{2}\cdot\frac{1-iy_a}{1+y_a^2}[JY, Y].$$

Proof. The proof is completely similar to that of Lemma 4.2, and omitted. □

Lemma 4.12. *If $h \in G_{a+1}(0)$ and $x \in \mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ with $j < a+1 \leqslant k$, then one has $Jhx = hJx$.*

Proof. Note that $hE_j = E_j$. Then the proof is entirely parallel to Lemma 4.1. □

For simplicity we put for every $j < a$

$$\mathfrak{q}_j^a := \sum_{k=a}^{r} (\mathfrak{n}_{(\alpha_k+\alpha_j)/2})_{\mathbb{C}}.$$

The adjoint action of $G_a(0)_{\mathbb{C}}$ leaves $\mathfrak{q}_j^a$ stable.

Proposition 4.13. *For every $a = 2, \ldots, r$, there is a constant $N_a > 0$ independent of $y \in V_a$ such that $\|\mathrm{Ad}_{\mathfrak{q}_j^a}(\eta_a(y)^{-1})\|_{\mathbf{s}} \leqslant N_a$ for $j < a$.*

Proof. We start with the case $a = r$. In this case we have $F_r = E_r$, $\mathfrak{g}_r(0) = \mathbb{R}H_r$, $V_r = \mathbb{R}E_r$ and $\mathfrak{q}_j^r = (\mathfrak{n}_{(\alpha_r+\alpha_j)/2})_{\mathbb{C}}$. Thus if we set $y = y_rE_r$ with $y_r \in \mathbb{R}$, then we have $\eta_r(y) = \exp\log(1+iy_r)H_r$. Hence the adjoint action of $\eta_r(y)^{-1}$ on the subspaces $\mathfrak{q}_j^r$ $(j < r)$ are equally the multiplication by the scalar $(1+iy_r)^{-1/2}$. Therefore the proposition is true for $a = r$.

Suppose that the proposition is true for the case $a+1$. Let us express every $y \in V_a$ as $y = y_aE_a + Y + y'$ with $Y \in \mathfrak{m}_a$ and $y' \in V_{a+1}$. Then we have the formula in Lemma 4.11. Let $\Omega_{a+1} := G_{a+1}(0)F_{a+1}$. It is a regular open convex cone in V_{a+1}. Since $[JY, Y] \in \overline{\Omega}_{a+1}$ by Lemma 4.4, there is a unique $h_{a+1} \in G_{a+1}(0)$

such that

$$h_{a+1}F_{a+1} = F_{a+1} + \frac{1}{2}\cdot\frac{1}{1+y_a^2}[JY, Y].$$

It is evident that h_{a+1} depends real analytically on y_a and Y. Then with

$$\tilde{y} := h_{a+1}^{-1}\left(y' - \frac{1}{2}\cdot\frac{y_a}{1+y_a^2}[JY, Y]\right)$$

we get

$$\bigl(\exp -i(1+iy_a)^{-1}JY\bigr)(F_a + iy) = \bigl(\exp\log(1+iy_a)H_a\bigr)h_{a+1}\eta_{a+1}(\tilde{y})F_a.$$

This implies that

$$\eta_a(y) = \bigl(\exp i(1+iy_a)^{-1}JY\bigr)\bigl(\exp\log(1+iy_a)H_a\bigr)h_{a+1}\eta_{a+1}(\tilde{y}). \tag{4.8}$$

Now we fix $j < a$ and express every element $Z \in \mathfrak{q}_j^a$ as $Z = Z_a + Z'$ with $Z_a \in (\mathfrak{n}_{(\alpha_a+\alpha_j)/2})_{\mathbb{C}}$ and $Z' \in \mathfrak{q}_j^{a+1}$. We have by using (2.3)

$$\bigl(\exp -i(1+iy_a)^{-1}JY\bigr)(Z_a + Z') = Z_a - i(1+iy_a)^{-1}[JY, Z_a] + Z'.$$

Since the action of $G_{a+1}(0)_{\mathbb{C}}$ on Z_a is trivial, (4.8) together with Lemma 4.12 yields

$$\begin{aligned}\eta_a(y)^{-1}(Z_a + Z') = {} & (1+iy_a)^{-1/2}Z_a - i(1+iy_a)^{-1}\eta_{a+1}(\tilde{y})^{-1}\bigl[Jh_{a+1}^{-1}Y, Z_a\bigr] \\ & + \eta_{a+1}(\tilde{y})^{-1}h_{a+1}^{-1}Z'.\end{aligned} \tag{4.9}$$

Now the definition of h_{a+1} leads us to

$$\frac{1}{2}\,\frac{1}{1+y_a^2}\,h_{a+1}^{-1}[JY, Y] = F_{a+1} - h_{a+1}^{-1}F_{a+1},$$

which together with Proposition 4.5 shows that there is a constant $N' > 0$ independent of y such that

$$\bigl\|h_{a+1}^{-1}[JY, Y]\bigr\|_{\mathbf{s}} \leqslant N'\bigl(1 + y_a^2\bigr). \tag{4.10}$$

Once we obtain this, we get just as in the proof of Lemma 4.9

$$\bigl\|h_{a+1}^{-1}Y\bigr\|_{\mathbf{s}}^2 \leqslant N'\|E_{\mathbf{s}}^*\|_{\mathbf{s}}\bigl(1 + y_a^2\bigr). \tag{4.11}$$

Now, since $[Jh_{a+1}^{-1}Y, Z_a] \in \mathfrak{q}_j^{a+1}$ and $h_{a+1}^{-1}Z' \in \mathfrak{q}_j^{a+1}$, induction hypothesis says that $\eta_{a+1}(\tilde{y})^{-1}$ has no contribution to the estimation of the right-hand side of (4.9). Therefore, in order to finish the proof, it only remains to estimate $h_{a+1}^{-1}Z'$. We put $\mathfrak{r}_j^a := \sum_{k=a}^{r}\mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ (real form of $\mathfrak{q}_j^a$) for simplicity in the following lemma. □

Lemma 4.14. *There is a positive constant N'' independent of $y \in V_a$ such that $\|\operatorname{Ad}_{\mathfrak{r}_j^{a+1}}(h_{a+1}^{-1})\|_{\mathbf{s}} \leqslant N''$ for $j < a$.*

Proof. Let us fix $j < a$. Since $[J\mathfrak{r}_j^{a+1}, \mathfrak{r}_j^{a+1}] \subset V_{a+1}$, Lemma 4.12 guarantees that the estimation method in the proof of Lemma 4.6 works well also in the present case. Thus $\|\operatorname{Ad}_{\mathfrak{r}_j^{a+1}}(h_{a+1}^{-1})\|_{\mathbf{s}}^2 \leqslant M_{\mathbf{s}}\|E_{\mathbf{s}}^*\|_{\mathbf{s}}\|\operatorname{Ad}_{V_{a+1}}(h_{a+1}^{-1})\|_{\mathbf{s}}$. Therefore, the lemma follows from Proposition 4.5. □

Since $\mathfrak{q}_1^2 = \mathfrak{m}_{\mathbb{C}}$, Proposition 4.13 for $a = 2$ gives the following corollary.

Corollary 4.15. *There is a constant $N_2 > 0$ independent of $y \in V_2$ such that $\| \mathrm{Ad}_{\mathfrak{m}_{\mathbb{C}}}(\eta_2(y)^{-1})\|_{\mathbf{s}} \leqslant N_2$.*

In the proof of Proposition 4.16 below, we use the notation of the previous subsection. Hence we have $E' = F_2$, $V' = V_2$. Moreover, we set $\eta' = \eta_2$, so that $\eta'(v)E' = E' + iv$ for every $v \in V'$.

Proposition 4.16. *One can find a positive constant $K_{\mathbf{s}}^1$ such that $\| \mathrm{Ad}_W(\eta(y)^{-1})\|_{\mathbf{s}} \leqslant K_{\mathbf{s}}^1$ for all $y \in V$.*

Proof. The proof is done by induction on the rank r of $\mathfrak{g}$. If $r = 1$, then $\mathrm{Ad}_W \eta(y)$ $(y \in \mathbb{R})$ is the multiplication by $1 + iy$, so that the proposition is true. Suppose now that the proposition holds for normal j-algebras of rank $r - 1$. Let $y \in V$, and we express y as $y = y_1E_1 + Y + y'$ with $y_1 \in \mathbb{R}$, $Y \in \mathfrak{m}$ and $y' \in V'$. Induction hypothesis implies that there is a positive constant $\widetilde{K}_{\mathbf{s}}$ such that

$$\left\|\mathrm{Ad}_{W'}(\eta'(v)^{-1})\right\|_{\mathbf{s}} \leqslant \widetilde{K}_{\mathbf{s}} \quad \text{for all } v \in V'. \tag{4.12}$$

In a way similar to getting the formula (4.8), we have

$$\eta(y) = \bigl(\exp i(1+iy_1)^{-1}JY\bigr)\bigl(\exp\log(1+iy_1)H_1\bigr)h_2\eta'(\tilde{y}), \tag{4.13}$$

with $h_2 \in G'(0)$ and $\tilde{y} \in V'$ given by

$$h_2E' = E' + \frac{1}{2}\cdot\frac{1}{1+y_1^2}[JY, Y],$$

$$\tilde{y} := h_2^{-1}\left(y' - \frac{1}{2}\cdot\frac{y_1}{1+y_1^2}[JY, Y]\right).$$

Now decompose W as $W = \mathbb{C}E_1 + \mathfrak{m}_{\mathbb{C}} + W'$ with $W' := V'_{\mathbb{C}}$. Let $z_1 \in \mathbb{C}$, $Z \in \mathfrak{m}_{\mathbb{C}}$ and $w' \in W'$. We have just in the same way as deriving (4.6)

$$\begin{aligned}
&\eta(y)^{-1}(z_1E_1 + Z + w') \\
&\quad = \frac{z_1}{1+iy_1}E_1 + \frac{1}{\sqrt{1+iy_1}}\bigl(\mathrm{Ad}_{\mathfrak{m}_{\mathbb{C}}}(\eta'(\tilde{y})^{-1}h_2^{-1})\bigr)\left(Z - \frac{iz_1}{1+iy_1}Y\right) \\
&\qquad + \bigl(\mathrm{Ad}_{W'}(\eta'(\tilde{y})^{-1}h_2^{-1})\bigr)\left(w' - \frac{i}{1+iy_1}[JY, Z] - \frac{z_1}{2(1+iy_1)^2}[JY, Y]\right).
\end{aligned}$$

Corollary 4.15 and (4.12) tell us that $\eta'(\tilde{y})^{-1}$ has no contribution to the estimate. Moreover, we have the estimates (4.10) and (4.11) for $a = 1$ by proceeding in the same way. Therefore Lemmas 4.1, 4.7 and Proposition 4.5 complete the induction. □

Now we are able to prove Theorem 4.10. Take $h_x \in G(0)$ so that $h_xE = E + x$. Then, with $y' := h_x^{-1}y \in V$, we get $\mathcal{I}_{\mathbf{s}}(x + E + iy) = h_x \cdot \mathcal{I}_{\mathbf{s}}(E + iy')$. Furthermore we have $\eta(y')E = E + iy'$. Therefore we get by virtue of (ii) of Lemma 3.10

$$\langle w, \mathcal{I}_{\mathbf{s}}(x + E + iy)\rangle = \bigl\langle \eta(y')^{-1}h_x^{-1}w, E_{\mathbf{s}}^*\bigr\rangle \quad (w \in W). \tag{4.14}$$

Propositions 4.5 and 4.16 tell us that the absolute value of the right-hand side of (4.14) is bounded by $K_{\mathbf{s}}^1K_{\mathbf{s}}\|w\|_{\mathbf{s}}\|E_{\mathbf{s}}^*\|_{\mathbf{s}}$. This completes the proof of Theorem 4.10.

4.4. Boundedness of Cayley images

We introduce a Hermitian inner product $(\cdot \mid \cdot)_{\mathbf{s}}$ on the complex vector space $U = (\mathfrak{g}(1/2), -J)$ by

$$(u_1 \mid u_2)_{\mathbf{s}} := \langle [Ju_1, u_2], E_{\mathbf{s}}^* \rangle - i\langle [u_1, u_2], E_{\mathbf{s}}^* \rangle \quad (u_1, u_2 \in U). \tag{4.15}$$

Then we have by (2.8) and (3.18)

$$(u_1 \mid u_2)_{\mathbf{s}} = 2\langle Q(u_1, u_2), E_{\mathbf{s}}^* \rangle = 2\langle Q(u_1, u_2) \mid E \rangle_{\mathbf{s}}. \tag{4.16}$$

Correspondingly we have a Hermitian inner product $(\cdot \mid \cdot)_{\mathbf{s}}$ and a norm $\|\cdot\|_{\mathbf{s}}$ on $U^\dagger$. Thus we obtain a Hermitian inner product and a norm, denoted by the same symbol, on $U^\dagger \oplus W^*$.

Theorem 4.17. *The Cayley image $\mathcal{C}_{\mathbf{s}}(D)$ of D is bounded for any* $\mathbf{s} > 0$.

Proof. First of all it is clear from (4.1) and Theorem 4.10 that the image $C_{\mathbf{s}}(\Omega + iV)$ of the tube domain $\Omega + iV$ is bounded in W^*. To proceed further, we take, given $x \in \overline{\Omega}$, the unique element $h_x \in G(0)$ such that $h_x E = E + x$, as before.

Lemma 4.18. *There is a positive constant $L_{\mathbf{s}}^0$ such that $\|\operatorname{Ad}_U h_x^{-1}\|_{\mathbf{s}} \leqslant L_{\mathbf{s}}^0$ holds for any $x \in \overline{\Omega}$.*

Proof. For simplicity we shall write $h_x^{-1}u$ instead of $(\operatorname{Ad}_U h_x^{-1})u$ for $u \in U$. By (4.16) we get for $u \in U$

$$\left\|h_x^{-1}u\right\|_{\mathbf{s}}^2 = 2\left\langle Q\left(h_x^{-1}u, h_x^{-1}u\right), E_{\mathbf{s}}^*\right\rangle = 2\left\langle h_x^{-1}Q(u,u), E_{\mathbf{s}}^*\right\rangle.$$

By Proposition 4.5, (2.8), (4.7) and (4.15), it holds that

$$2\left\|h_x^{-1}Q(u,u)\right\|_{\mathbf{s}} \leqslant 2\left\|\operatorname{Ad}_V h_x^{-1}\right\|_{\mathbf{s}}\|Q(u,u)\|_{\mathbf{s}} \leqslant K_{\mathbf{s}}M_{\mathbf{s}}\|u\|_{\mathbf{s}}^2.$$

Clearly this proves the lemma. □

Let us continue the proof of Theorem 4.17. Let $z = (u, w) \in D$ and put $w = x + iy$. By (2.9), we have $x \in \frac{1}{2}Q(u,u) + \Omega \subset \Omega$. Therefore by the above, $\|C_{\mathbf{s}}(w)\|_{\mathbf{s}}$ is bounded by a positive constant independent of z. In view of (4.2), it remains to show that the norm of the antilinear form $u' \mapsto \langle Q(u,u'), \mathcal{I}_{\mathbf{s}}(w+E)\rangle$ on U is bounded by a positive constant independent of z. Let $h_x \in G(0)$ be as above, so that we have $h_x E = x + E$. Then with $y' := h_x^{-1}y \in V$ we get $\mathcal{I}_{\mathbf{s}}(w+E) = h_x \cdot \mathcal{I}_{\mathbf{s}}(E + iy')$. Hence

$$\langle Q(u,u'), \mathcal{I}_{\mathbf{s}}(w+E)\rangle = \left\langle Q\left(h_x^{-1}u, h_x^{-1}u'\right), \mathcal{I}_{\mathbf{s}}(E+iy')\right\rangle.$$

We know $\|\operatorname{Ad}_U h_x^{-1}\|_{\mathbf{s}} \leqslant L_{\mathbf{s}}^0$ by Lemma 4.18 and $\|\mathcal{I}_{\mathbf{s}}(E+iy')\|_{\mathbf{s}} \leqslant L_{\mathbf{s}}$ by Theorem 4.10. Thus our only task is to estimate $\|h_x^{-1}u\|_{\mathbf{s}}$. Now (4.16) gives

$$\left\|h_x^{-1}u\right\|_{\mathbf{s}}^2 = 2\left\langle Q\left(h_x^{-1}u, h_x^{-1}u\right), E_{\mathbf{s}}^*\right\rangle = 2\left\langle Q(u,u),\ h_x \cdot E_{\mathbf{s}}^*\right\rangle \leqslant 4\left\langle x,\ h_x \cdot E_{\mathbf{s}}^*\right\rangle,$$

because $2x - Q(u,u) \in \Omega$ and $h_x \cdot E_{\mathbf{s}}^* \in \Omega^*$. Here, since $h_x^{-1}x = E - h_x^{-1}E$, we have

$$\langle x,\ h_x \cdot E_{\mathbf{s}}^*\rangle = \left\langle E - h_x^{-1}E, E_{\mathbf{s}}^*\right\rangle \leqslant \langle E, E_{\mathbf{s}}^*\rangle = \|E\|_{\mathbf{s}}^2.$$

This implies $\|h_x^{-1}u\|_{\mathbf{s}} \leqslant 2\|E\|_{\mathbf{s}}$, completing the proof. □

4.5. *Inverse map of Cayley transform*

For every $w \in W$, we introduce a complex linear operator $\varphi_{\mathbf{s}}(w)$ on U by the formula

$$(\varphi_{\mathbf{s}}(w)u_1 \mid u_2)_{\mathbf{s}} = 2\langle Q(u_1, u_2) \mid w\rangle_{\mathbf{s}} \quad (u_1, u_2 \in U). \tag{4.17}$$

It is obvious from (4.16) that $\varphi_{\mathbf{s}}(E)$ is the identity operator on U. Extending $\iota_{\mathbf{s}}$ defined by (3.17) canonically to a complex linear bijection from W^* to W, we have for any $f \in W^*$

$$\langle Q(u, u'), f\rangle = \langle Q(u, u') \mid \iota_{\mathbf{s}}(f)\rangle_{\mathbf{s}} = \frac{1}{2}\bigl(\varphi_{\mathbf{s}}\bigl(\iota_{\mathbf{s}}(f)\bigr)u \mid u'\bigr)_{\mathbf{s}} \quad (u, u' \in U).$$

Therefore for every $F \in U^{\dagger}$, defining $\iota_{\mathbf{s}}(F) \in U$ by

$$\langle u, F\rangle = \bigl(\iota_{\mathbf{s}}(F) \mid u\bigr)_{\mathbf{s}} \quad (u \in U),$$

we obtain $\iota_{\mathbf{s}}(\langle Q(u, \cdot), f\rangle) = \frac{1}{2}\varphi_{\mathbf{s}}(\iota_{\mathbf{s}}(f))u$. Hence our Cayley transform $\mathcal{C}_{\mathbf{s}}(u, w)$ in (4.2) is rewritten as

$$\mathcal{C}_{\mathbf{s}}(u, w) = \Bigl(\iota_{\mathbf{s}}^{-1}\bigl(\varphi_{\mathbf{s}}\bigl(\iota_{\mathbf{s}}(\mathcal{I}_{\mathbf{s}}(w + E))\bigr)u\bigr),\ C_{\mathbf{s}}(w)\Bigr). \tag{4.18}$$

A direct computation yields

Proposition 4.19. *One has*

$$C_{\mathbf{s}}^{-1}(f) = 2\mathcal{I}_{\mathbf{s}}^*(E_{\mathbf{s}}^* - f) - E \quad (f \in W^*),$$
$$\mathcal{C}_{\mathbf{s}}^{-1}(F, f) = \Bigl(2\varphi_{\mathbf{s}}\bigl(E - \iota_{\mathbf{s}}(f)\bigr)^{-1}\bigl(\iota_{\mathbf{s}}(F)\bigr),\ C_{\mathbf{s}}^{-1}(f)\Bigr) \quad \bigl((F, f) \in U^{\dagger} \times W^*\bigr).$$

Theorem 4.20. *The Cayley transform $\mathcal{C}_{\mathbf{s}}$ is a birational map which sends the Siegel domain D biholomorphically onto the bounded domain $\mathcal{C}_{\mathbf{s}}(D)$.*

Proof. The birationality follows from Proposition 4.19. The rest of the proof is completely similar to that of [10, Theorem 3.6]. □

5. Non-symmetric 4-dimensional Siegel domain

Here we take a close look at the **s**-Cayley transform of Pjatetskii-Shapiro's non-symmetric complex 4-dimensional Siegel domain, see [15, p. 26].

Let $V = \mathrm{Sym}(2, \mathbb{R})$, the real vector space of 2×2 real symmetric matrices. In V we have the open convex cone Ω of positive definite matrices. The complexification $W = V_{\mathbb{C}}$ is the space $\mathrm{Sym}(2, \mathbb{C})$ of 2×2 complex symmetric matrices. We put $U = \mathbb{C}$, and write $U_{\mathbb{R}}$ when we regard U as $\mathbb{R}^2$ canonically. Consider the Hermitian sesqui-linear map $Q : U \times U \to W$ defined by

$$Q(u_1, u_2) := 2\begin{pmatrix} 0 & 0 \\ 0 & u_1\bar{u}_2 \end{pmatrix}.$$

It is clear that Q is Ω-positive. With these data Pjatetskii-Shapiro's Siegel domain D is now defined by (2.9). By identifying $w = \left(\begin{smallmatrix} w_1 & w_2 \\ w_2 & w_3 \end{smallmatrix}\right) \in \mathrm{Sym}(2, \mathbb{C})$ with $(w_1, w_2, w_3) \in \mathbb{C}^3$, it is also described as

$$D = \bigl\{(u, w_1, w_2, w_3) \in \mathbb{C}^4;\ v_1\bigl(v_3 - |u|^2\bigr) - v_2^2 > 0,\ v_3 > |u|^2\bigr\} \quad (v_j = \mathrm{Re}\, w_j).$$

Since Ω is irreducible, D is irreducible. The Lie group

$$H = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix};\ a > 0,\ c > 0,\ b \in \mathbb{R} \right\}$$

acts on Ω simply transitively by $\rho(h)v = h\, v\, {}^{t}h$. Let σ be the positive character $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \mapsto c$ of H. It is evident that

$$\rho(h) Q(u_1, u_2) = Q\big(\sigma(h)u_1, \sigma(h)u_2\big).$$

Let N_D be the 2-step nilpotent Lie group with group law (2.10) for $a \in V$ and $b \in U_{\mathbb{R}}$. Then we see that the action of H on N_D given by $h \cdot n(a,b) = n(\rho(h)a, \sigma(h)b)$ is in fact a homomorphism of H into the automorphism group $\mathrm{Aut}(N_D)$ of N_D. The group $G = N_D \ltimes H$ acts on D simply transitively.

Let us describe the normal j-algebra structure of $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{h} \oplus U_{\mathbb{R}} \oplus V$, where $\mathfrak{h} = \mathrm{Lie}(H)$. Define a linear map J on $\mathfrak{g}$ by

$$J : V \ni v = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1/2 & 0 \\ v_2 & v_3/2 \end{pmatrix} \in \mathfrak{h}, \qquad Ju = iu\ (u \in U),$$
$$J : \mathfrak{h} \ni \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \mapsto -\begin{pmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{pmatrix} \in V.$$

Then with $\omega = 0 \oplus 0 \oplus \mathrm{tr} \in \mathfrak{h}^* \oplus (U_{\mathbb{R}})^* \oplus V^*$, the triple $(\mathfrak{g}, J, \omega)$ is a normal j-algebra. Let e_{ij} be the 2×2 matrix unit with the only non-zero (i,j)-entry equal to 1. Set

$$H_j := \tfrac{1}{2} e_{jj} \quad (j = 1,2), \qquad T_{21} := e_{21}, \qquad E_k := e_{kk} \quad (k = 1,2), \qquad X_{21} := e_{12} + e_{21}.$$

These 6 elements together with the canonical basis of $U_{\mathbb{R}} = \mathbb{R}^2$ form a linear basis of $\mathfrak{g}$. We put $\mathfrak{a} := \mathbb{R}H_1 \oplus \mathbb{R}H_2$. Let $\alpha_1, \alpha_2 \in \mathfrak{a}^*$ be the basis dual to H_1, H_2, that is, we have $\alpha_k(H_j) = \delta_{kj}$. It is easy to see that the root spaces are

$$\mathfrak{n}_{(\alpha_2 - \alpha_1)/2} = \mathbb{R}T_{21}, \qquad \mathfrak{n}_{(\alpha_2 + \alpha_1)/2} = \mathbb{R}X_{21}, \qquad \mathfrak{n}_{\alpha_k} = \mathbb{R}E_k \quad (k = 1,2), \qquad \mathfrak{n}_{\alpha_2/2} = U_{\mathbb{R}}.$$

Note that the root $\alpha_1/2$ is missing. In particular, we see that D is not quasisymmetric by [1, Proposition 3]. The constants in (2.7) are

$$n_{21} = 1, \qquad d_1 = d_2 = 3/2, \qquad b_1 = 0, \qquad b_2 = 1. \tag{5.1}$$

Let $\mathbf{s} = (s_1, s_2) \in \mathbb{R}^2$ and suppose $\mathbf{s} > 0$. The inner product $\langle \cdot \mid \cdot \rangle_{\mathbf{s}}$ in (3.3) restricted to V is described as

$$\langle v \mid v' \rangle_{\mathbf{s}} = s_1 v_1 v_1' + 2 s_2 v_2 v_2' + s_2 v_3 v_3'.$$

The Hermitian inner product $(u \mid u')_{\mathbf{s}}$ of U is equal to $4 s_2 u \bar{u}'$. Therefore the linear map $\varphi_{\mathbf{s}}(w)$ $(w \in W)$ defined by (4.17) is the scalar multiplication operator by w_3 (independent of $\mathbf{s}$).

Let us write down the pseudoinverse map $\mathcal{I}_{\mathbf{s}}$. The details being left to the reader, we have for $y = \left(\begin{smallmatrix} y_1 & y_2 \\ y_2 & y_3 \end{smallmatrix}\right) \in \Omega$

$$\iota_{\mathbf{s}}\big(\mathcal{I}_{\mathbf{s}}(y)\big) = \frac{1}{\det y} \begin{pmatrix} y_3 - s_1^{-1}(s_1 - s_2) y_1^{-1} y_2^2 & -y_2 \\ -y_2 & y_1 \end{pmatrix}, \tag{5.2}$$

where $\iota_{\mathbf{s}}$ is as in (3.17). Put $\Omega^{\mathbf{s}} := \iota_{\mathbf{s}}(\Omega^*) \subset V$. Then

$$\Omega^{\mathbf{s}} = \left\{ \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \in V;\ y_1 y_3 - \frac{s_2}{s_1} y_2^2 > 0,\ y_3 > 0 \right\},$$

and the dual pseudoinverse map $\mathcal{I}_{\mathbf{s}}^* : \Omega^{\mathbf{s}} \to \Omega$ is given by

$$\mathcal{I}_{\mathbf{s}}^*(y) = \frac{1}{y_1 y_3 - s_1^{-1} s_2 y_2^2} \begin{pmatrix} y_3 & -y_2 \\ -y_2 & y_1 + s_1^{-1}(s_1 - s_2) y_2^2 y_3^{-1} \end{pmatrix}.$$

Now by (5.2) we have

$$\varphi_{\mathbf{s}}\big(\iota_{\mathbf{s}}\big(\mathcal{I}_{\mathbf{s}}(w)\big)\big) = \text{mutiplication by } \frac{w_1}{\det w},$$

which is not equal to $\varphi_{\mathbf{s}}(w)^{-1}$ unless w is a diagonal matrix. In this way, the Cayley transform $\mathcal{C}_{\mathbf{s}}(u, w)$ is described as (cf. (4.18))

$$\iota_{\mathbf{s}}\big(\mathcal{C}_{\mathbf{s}}(u, w)\big) = \left(\frac{w_1 + 1}{\det(w + E)} u,\ E - 2\,\iota_{\mathbf{s}}\big(\mathcal{I}_{\mathbf{s}}(w + E)\big)\right). \tag{5.3}$$

If $s_1 = s_2$, then (5.2) yields $\iota_{\mathbf{s}}(\mathcal{I}_{\mathbf{s}}(y)) = y^{-1}$, so that the right-hand side of (5.3) coincides with Geatti's Cayley transform Φ_2 [6, Proposizione 2.1] up to a trivial modification to the present context. However, note that Remark 3.5 and (5.1) imply that the Koszul form β in the present case coincides with $3E_1^* + 4E_2^*$ on V. Therefore we see that Geatti's Cayley transform is not the Cayley transform considered in [10] and [11]. Though Geatti's might be regarded as the most natural one in view of (5.3), it does not fit to the problems treated in [11–13]. This mismatch reflects the common understanding that there is no canonical bounded model for non-quasisymmetric Siegel domains.

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